UNIT-04

09/09/2025 **NORMAL DISTRIBUTIONS** Tuesday

Definition: Gaussian distribution, how the values of a random variable are distributed, with most observations clustering around the mean and fewer appearing as you move away from it.

A continuous random variable X in the interval $(-\infty, \infty)$ with probability density function.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$
$$\sigma > 0$$

is called normal random variable with the parameters μ and σ^2. It is usually denoted by $X \sim N(\mu, \sigma^2)$, where μ is mean and σ^2 is variance of normal distribution. The total area under the curve is 1

Heights, weights and exam scores of a large population often follow a normal distribution

*Normal Integrals:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma\sqrt{2\pi}$$

* properties of Normal distribution:

1. The probability curve of the Normal distribution is bell-shaped and Symmetrical about $x=\mu$

2. Mean, Median and Mode of the normal distribution coincide (i.e. equal)

3. The maximum probability occurs at the point $x=\mu$, the p.d.f at the point $x=\mu$ is $[f(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$

4. Normal distribution is a Symmetrical distribution i.e. $\beta_1 = 0$ and $\gamma_1 = 0$

5. Normal distribution is mesokurtic distribution i.e., $\beta_2 = 3$ and $\gamma_2 = 0$

6. All odd order moments of normal distribution vanish i.e., zero $\mu_{2n+1} = 0$, $n = 0, 1, 2, \dots$

7. Even order moments of normal distribution are $\mu_{2n} = 1 \cdot 3 \cdot 5 \dots (2n-1)\,\sigma^{2n}$, $n = 0, 1, 2, \dots$

8. linear combination of independent normal variate is also a normal variate.

9. Normal distribution curve is asymptotic to x-axis.

10. The points of inflexion of the normal distribution are $\mu \pm \sigma$, and p.d.f at point of inflexion is.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$$

11. The quartile deviation, mean deviation and standard deviation of the Normal distribution are $2/3\sigma$, $4/5\sigma$ and σ respectively.

12. The ratio of Q.D, M.D and SD is 10:12:15.

13. The M.g.f of the Normal distribution is
$$M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$$

14. Cumulant generating function of the Normal distribution is

$$K_x(t) = t\mu + \frac{t^2\sigma^2}{2}$$

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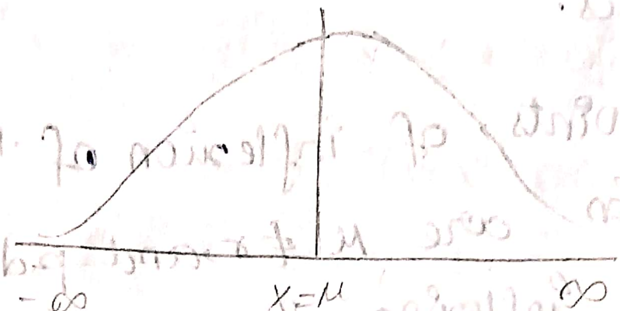
15. The characteristic function of Normal distribution is $\phi_X(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}$.

16. The Normal probability curve is



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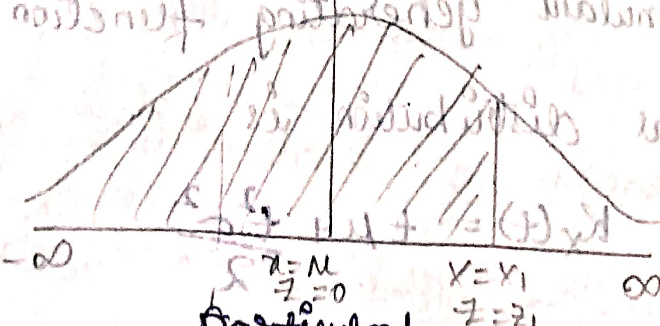
17. Area's property of normal distribution:

If $X \sim N(\mu, \sigma^2)$, then the probability of the normal variate X lies between $-\infty$ and x_1 is $P(-\infty < X < x_1) = P\left[\frac{-\infty-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{x_1-\mu}{\sigma}\right] = P(-\infty < Z < z_1) = \int_{-\infty}^{z_1} \phi(z)\,dz$

this can be shown in the fig and obtained from standard normal tables from different values of z. If z_1 is positive.



Particularly

(i) $P(\mu-\sigma < X < \mu+\sigma) = 0.6826$

(ii) $P(\mu-2\sigma < X < \mu+2\sigma) = 0.9544$

(iii) $P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$

* M.G.F of Normal distribution:

$$M_x(t) = E(e^{tx})$$

$$= \int e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x-\mu}{\sigma}$

$x = \mu + \sigma z$

$dx = \sigma\, dz$

Limits if $x \to -\infty$; $z \to -\infty$

if $x \to \infty$; $z \to \infty$

$$M_x(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + t\sigma z} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2 + \frac{t^2\sigma^2}{2}} dz$$

$$= \frac{e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2} dz$$

let $z - t\sigma = u$

$dz = du$

$$M_X(t) = e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$ [∵ from standard normal integral]

$$\boxed{M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}}$$ //

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*Characteristic function of ~~the~~ Normal distribution:

$$\phi_X(t) = E(e^{itx})$$

$$= \int e^{itx} \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x-\mu}{\sigma}$

$x = \mu + \sigma z$

$dx = \sigma dz$

limits $-\infty < z < \infty$

$$\phi_x(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it(\mu+\sigma z)} \cdot e^{-\frac{z^2}{2}} \cancel{\sigma} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2 + it\sigma z} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2it\sigma z)} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2it\sigma z - i^2t^2\sigma^2 + i^2t^2\sigma^2)} dz$$

$+t^2\sigma^2 \quad -t^2\sigma^2$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - it\sigma)^2} \cdot e^{-\frac{t^2\sigma^2}{2}} dz$$

$$= \frac{e^{it\mu - \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - it\sigma)^2} dz$$

let $z - it\sigma = u$ $\qquad \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$

$dz = du$

$-\infty < u < \infty$

$$\phi_x(t) = \frac{e^{it\mu - \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= \frac{e^{itu - \frac{t^2\sigma^2}{2}} \cdot \sqrt{2\pi}}{\sqrt{2\pi}}$$

$$\boxed{\phi_x(t) = e^{itu - \frac{t^2\sigma^2}{2}}}$$

* Cummulative Generating function of Normal distribution:

$$K_x(t) = \log M_x(t)$$

$$= \log e^{\left(t\mu + \frac{t^2\sigma^2}{2}\right)} = t\mu + \frac{t^2\sigma^2}{2}$$

Cummulants and moments

$K_1 = \text{Mean} = \mu_1' = \text{Coefficient of } t \text{ in } K_x(t) = \mu$

$K_2 = \mu_2 = \text{Variance} = \text{Coefficient of } \frac{t^2}{2!} \text{ in } K_x(t)$

$= \sigma^2$

$K_3 = \mu_3 = \text{Coefficient of } \frac{t^3}{3!} \text{ in } K_x(t) = 0$

$K_4 = \text{Coefficient of } \frac{t^4}{4!} \text{ in } K_x(t) = 0$

$\mu_4 = K_4 + 3K_2^2 = 0 + 3(\sigma^2)^2 = 3\sigma^4$

* Skewness of ND:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \quad \gamma_1 = \sqrt{\beta_1} = 0$$

Normal distribution is a Symmetrical distribution

* Kurtosis of ND:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{(\sigma^2)^2} = 3$$

$$\gamma_2 = \beta_2 - 3 = 3-3 = 0$$

Normal distribution is a mesokurtic distribution.

* Odd order moments about mean of Normal distribution:

$$\mu_{2n+1} = E\left[[X-\text{Mean}]^{2n+1}\right] = E\left[(X-\mu)^{2n+1}\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow x-\mu = z\sigma$

$dx = \sigma dz$

Limits $-\infty < z < \infty$

$$\mu_{2n+1} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} \cdot e^{-z^2/2} \sigma\, dz$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} \cdot e^{-z^2/2} \, dz = 0$$

Since $z^{2n+1} e^{-z^2/2}$ is an odd function

$\therefore \mu_{2n+1} = 0 \; ; \; n = 0, 1, 2, 3, \ldots$

All odd order moments about mean are 0 i.e, vanish

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* Even order moments about mean and recurrence relation for moments of Normal distribution:-

$$\mu_{2n} = E\left[(x-\text{mean})^{2n}\right] = E\left[(x-\mu)^{2n}\right]$$

$$= \int (x-\mu)^{2n} f(x) dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $\frac{x-\mu}{\sigma} = z \Rightarrow x - \mu = z\sigma$

$x = \mu + z\sigma$

$dx = \sigma dz$

limits $-\infty < z < \infty$

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} e^{-z^2/2} \sigma \, dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-z^2/2} \, dz \qquad \left[\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx \right]$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} 2 \cdot \int_0^{\infty} z^{2n} e^{-z^2/2} \, dz$$

since $z^{2n} e^{-z^2/2}$ is an even function

let $t = \frac{z^2}{2} \Rightarrow dt = \frac{2z\,dz}{2}$

$\Rightarrow dt = z\,dz \Rightarrow dz = \frac{dt}{z}$ $\left[\because t = \frac{z^2}{2} \Rightarrow z^2 = 2t \right.$

$dz = \frac{dt}{\sqrt{2t}}$ $\left. z = \sqrt{2t} \right]$

let $0 < t < \infty$

$$\mu_{2n} = \frac{\sigma^{2n}}{\sqrt{2\pi}} \cdot 2\int_0^{\infty} (2t)^n \cdot e^{-t} \frac{dt}{\sqrt{2t}} \qquad \frac{dt}{\sqrt{t}} = dt \times t^{-1/2}$$

$t^{n-1/2}$

$$= \frac{\sigma^{2n} 2^n}{\sqrt{\pi}} \cdot \int_0^{\infty} e^{-t} t^{n-\frac{1}{2}} dt$$

$$= \frac{\sigma^{2n} \cdot 2^n}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{n-\frac{1}{2}+1-1} dt$$

$$= \frac{\sigma^{2n} \cdot 2^n}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\left(n+\frac{1}{2}\right)-1} dt$$

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$$\mu_{2n} = \frac{\sigma^{2n} \cdot 2^{n}}{\sqrt{\pi}} \Gamma\left(n+\tfrac{1}{2}\right) \quad — ①$$

Replace n by $n-1$ in eq ①

$$\mu_{2n-2} = \frac{\sigma^{2n-2} \cdot 2^{n-1}}{\sqrt{\pi}} \Gamma\left(n-\tfrac{1}{2}\right) \quad — ②$$

divide eq ① by eq ② we get

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \frac{\sigma^{2n} \cdot 2^{n}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}\right)} \frac{\sqrt{\pi}}{\sigma^{2n-2}2^{n-1}}$$

$$= 2\sigma^2 \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}\right)}$$

$$= 2\sigma^2 \frac{\Gamma\left(n+\frac{1}{2}-1+1\right)}{\Gamma\left(n-\frac{1}{2}\right)}$$

$$= 2\sigma^2 \frac{\Gamma\left(n-\frac{1}{2}+1\right)}{\Gamma\left(n-\frac{1}{2}\right)}$$

$$= 2\sigma^2 \frac{\left(n-\frac{1}{2}\right)\cancel{\Gamma\left(n-\frac{1}{2}\right)}}{\cancel{\Gamma\left(n-\frac{1}{2}\right)}}$$

$$= 2\sigma^2 \frac{(2n-1)}{2}$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = (2n-1)\sigma^2$$

$$\mu_{2n} = (2n-1)\sigma^2 \cdot \mu_{2n-2} \quad \text{—— (3)}$$

This is called recurrence relation for the moments of Normal distribution.

In the eq (3),

Replace

n by $n-1$, $\mu_{2n-2} = (2n-3)\sigma^2 \mu_{2n-4}$

n by $n-2$, $\mu_{2n-4} = (2n-5)\sigma^2 \mu_{2n-6}$

$\vdots$

n by 2, $\mu_4 = 3 \cdot \sigma^2 \cdot \mu_2$

n by 1, $\mu_2 = 1 \cdot \sigma^2 \mu_0 = \sigma^2$ $\quad [\because \mu_0 = 1]$

from eq (3)

$$\mu_{2n} = (2n-1)\sigma^2 \mu_{2n-2}$$

$$= (2n-1)\sigma^2 (2n-3)\sigma^2 \mu_{2n-4}$$

$\vdots$

$$= (2n-1)\sigma^2 (2n-3)\sigma^2 (2n-5)\sigma^2 \cdots 3\sigma^2 \cdot 1 \cdot \sigma^2$$

$$= 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)(\sigma^2)^n$$

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$$M_{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$$

* Importance of Normal distribution:

Normal distribution has many no. of applications in statistics, some are listed below:

1. The discrete probability distributions viz., binomial, poisson, Hypergeometric etc., can be approximated to normal distribution for large values.
2. All small sample distributions viz., chi-square, t, F and Z, approximated to normal distribution for large values
3. The entire small sample theory is based on the fundamental assumption that the parent population from which the sample is drawn is assumed to be normal.
4. In conduction of large sample tests, the sampling distribution of sample means, sample variances, sample proportions etc.

tends to normal distribution.

5. Entire large sample theory depends on the area's property.

6. The theory of statistical quality control depends on 3σ limits developed by using

$$P(|Z| \le 3) = 0.9973$$

(or) $P(|Z| > 3) = 0.0027$

7. Since error function follows normal distribution $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, it has a fundamental importance in the theory of errors in Astronomy.

12/09/2025 Friday

* **Computation of mean, median and mode of Normal distribution:-**

Mean of N.D:

w.k.t c.g.f of ND is

$$K_X(t) = t\mu + \frac{t^2\sigma^2}{2}$$

$K_1 = \mu_1' = \text{mean} = \text{coefficient of } t \text{ in } K_X(t) = \mu$

$\therefore$ Mean of ND is μ

Median of N.D:

If M is the median, it can be calculated by

$p(X<M) = (X>M) = 1/2$

consider $p(X<M) = 1/2$

$$\int_{-\infty}^{M} f(x)\,dx = 1/2$$

$$\int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1/2$$

$$\int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx =$$

$$\Rightarrow \frac{1}{2} + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \quad \left[\because \int_{-\infty}^{\mu} f(x)\,dx = 1/2\right.$$ from the area's property

$$\Rightarrow \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

Mode of N.D:

Mode is the value of x for which $f(x)$ is maximum by using principle of maxima and minima.

$$f'(x) = 0 \text{ and } f''(x) < 0$$

we know that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \; ; -\infty < x < \infty$$

Consider

$$\log f(x) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

diff w.r.t x

$$\frac{1}{f(x)} = f'(x) = \frac{1}{2\sigma^2} = 2(x-\mu)$$

$$\frac{f'(x)}{f''(x)} = \frac{-1}{\sigma^2}(x-\mu)$$

$$f'(x) = \frac{-1}{\sigma^2}(x-\mu)\, f(x).$$

$$f''(x) = \frac{-1}{\sigma^2}\left\{(x-\mu)\, f'(x) + f(x) \cdot 1\right\}$$

$$= \frac{-1}{\sigma^2}\left\{(x-\mu)\left[\frac{-(x-\mu)}{\sigma^2}\right] f(x) + f(x)\right\}$$

$$f''(x) = \frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] \quad \text{— (2)}$$

$$f'(x) = 0$$

$$\frac{-1}{\sigma^2}(x-\mu)\, f(x) = 0$$

$$x-\mu = 0$$

$$x = \mu$$

from eq (2) $f''(x) = \frac{-f(x)}{\sigma^2}(1-0) \Rightarrow \frac{-f(x)}{\sigma^2} < 0$

Mode of ND is μ

Mean, Median, mode of Normal distribution are equal to μ

* Reproductive & additive property of Normal distribution:

Statement: Sum of "n" independent normal variates is also a normal variate

proof: let $x_1, x_2, \ldots x_n$ be n independent normal variates with mean's $\mu_1, \mu_2 \ldots \mu_n$ and variance's $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$ respectively.

The M.G.F are

$M_{x_1}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}$, $M_{x_2}(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$, $\ldots M_{x_n}(t) = e^{t\mu_n + \frac{t^2\sigma_n^2}{2}}$

Consider M.G.F of $x_1 + x_2 + \ldots x_n$ as

$$M_{x_1 + x_2 + \ldots x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \ldots M_{x_n}(t)$$

$$= e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}} \cdot e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}} \ldots e^{t\mu_n + \frac{t^2\sigma_n^2}{2}}$$

$$= e^{t(\mu_1 + \mu_2 + \ldots \mu_n) + \frac{t^2(\sigma_1^2 + \sigma_2^2 + \ldots \sigma_n^2)}{2}}$$

Which is looking to be a M.G.F of Normal distribution,

Hence, By uniqueness theorem of M.G.F, the distribution of sum of n normal variates i.e., $X_1 + X_2 \dots X_n$ is also a normal variate with mean's $\mu_1 + \mu_2 \dots \mu_n$ and variance's $\sigma_1^2 + \sigma_2^2 + \dots \sigma_n^2$.

Hence proved.

** **Difference of two normal variates:**

Let X, Y be two independent normal variates with mean's μ_1, μ_2 and variance's σ_1^2 and σ_2^2 respectively, then the difference of normal variates i.e., $X-Y$ is also a normal variate with mean's $\mu_1 - \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$ i.e., $X-Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.

* **Linear combination of Normal variates:**

Statement: Linear combination of "n" normal variates is also a normal variate.

Proof: Let $X_1, X_2, \dots X_n$ be "n" independent normal variates with mean $\mu_1, \mu_2 \dots \mu_n$ and variance $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$.

Consider,

$M_{X_1}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}}, M_{X_2}(t) = e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}, \dots$

$M_{X_n}(t) = e^{t\mu_n + \frac{t^2\sigma_n^2}{2}}$

If $a_1, a_2 \ldots a_n$ are constants then consider

$M_{a_1x_1 + a_2x_2 + \cdots a_nx_n}(t) = M_{a_1x_1}(t) \cdot M_{a_2x_2}(t) \ldots$

$M_{a_nx_n}(t)$

$= M_{x_1}(a_1t)\, M_{x_2}(a_2t) \ldots M_{x_n}(a_nt)$

$= e^{a_1t\mu_1 + \frac{a_1^2t^2\sigma_1^2}{2}} \cdot e^{a_2t\mu_2 + \frac{a_2^2t^2\sigma_2^2}{2}} \ldots$

$e^{a_nt\mu_n + \frac{a_n^2t^2\sigma_n^2}{2}}$

which is looking to be

$= e^{t(a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n) + \frac{t^2(a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots a_n^2\sigma_n^2)}{2}}$

which is looking to be the mgf of normal distribution.

Hence by uniqueness theorem of m.g.f of linear combination of 'n' independent normal variates is also a normal variate with mean $a_1\mu_1 + a_2\mu_2 + \cdots a_n\mu_n$ and variance $a_1\sigma_1^2 + a_2\sigma_2^2 + \cdots a_n\sigma_n^2$

i.e., $\sum_{i=1}^{n} a_ix_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

Remarks:

1. If $a_1 = 1$, $a_2 = 1$, $a_3 = a_4 = \cdots = a_n = 0$

then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

2. If $a_1 = 1$, $a_2 = -1$, $a_3 = a_4 = \dots = a_n = 0$ then

$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

3. If $a_1 = a_2 \dots = a_n = 1$ then

$\sum_{i=1}^{n} x_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$

4. If $x_i \sim N(\mu, \sigma^2)$ and $a_1 = a_2 = \dots a_n = 1$

then $\sum_{i=1}^{n} x_i \sim N(n\mu, n\sigma^2)$ and

$\frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

* **points of inflection of Normal probability curve:**

points of inflexion of any curve can be obtained by $f''(x)=0$ and $f'''(x) \neq 0$

For any normal curve

$$f''(x) = -\frac{f(x)}{\sigma^2}\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$f''(x)=0 \Rightarrow -\frac{f(x)}{\sigma^2}\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]=0,$$

$$1-\left(\frac{x-\mu}{\sigma}\right)^2=0 \qquad \left(\frac{x-\mu}{\sigma}\right)^2=1$$

$$(x-\mu)^2=\sigma^2 \qquad x-\mu=\pm\sigma$$

$$x=\mu\pm\sigma$$

$$f'''(x) = -\frac{f(x)}{\sigma^2}\left[-\frac{2(x-\mu)}{\sigma^2}\right]+\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]\left(-\frac{f'(x)}{\sigma^2}\right)$$

$$= \frac{2f(x)(x-\mu)}{\sigma^4} - \frac{f'(x)}{\sigma^2}\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

at $x=\mu\pm\sigma$ $f'''(x) = \frac{2f(x)}{\sigma^3} - f'(x)[1-] = \frac{2f(x)}{\sigma^3} \neq 0$

The points of inflexion of normal probability curve is $\mu\pm\sigma$.

$\mu-\sigma$ and $\mu+\sigma$ are equidistant from the mean μ.

The pdf of normal distribution at points of inflexion is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$

S.M. Shamim

4th lesson:

Normal distribution of mean and variances

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$

$x - \mu = \sigma z$

D.w.r.to x

$dx = \sigma dz$

$x = -\infty, z = -\infty$

$x = \infty, z = \infty$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-1/2 z^2} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \sigma \underbrace{\int_{-\infty}^{\infty} z e^{-1/2 z^2} dz}_{O.F} + \frac{\mu}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-1/2 z^2} dz}_{E.F}$$

$$\cancel{= 0 + \frac{\mu}{\sqrt{2\pi}} 2 \int_{0}^{\infty}}$$

even function

$f(-x) = f(x)$

$\int_{-\infty}^{\infty} f(x) dx = 2\int_{0}^{\infty} f(x) dx$

$$= 0 + \frac{\mu}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-1/2 z^2} dz$$

$$= \frac{2\mu}{\cancel{\sqrt{2\pi}}} \cancel{\sqrt{\frac{\pi}{2}}}$$

odd function

$f(-x) = -f(x)$

$\int_{-\infty}^{\infty} f(x) dx = 0$

$$\boxed{E(x) = \mu \quad = \text{Mean}}$$

$\int_{0}^{\infty} e^{-1/2 z^2} dz = \sqrt{\frac{\pi}{2}}$

$$\text{Var}(x) = E[(x-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^2 e^{-1/2\, z^2} \cancel{\sigma}\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-1/2\, z^2} dz$$

E.F

$$= \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^{\infty} z^2 e^{-1/2\, z^2} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^{\infty} 2t\, e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{\sigma^2 2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{1/2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{3/2-1} dt \qquad = 1+\frac{1}{2}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(3/2\right) = \frac{\cancel{2}\sigma^2}{\sqrt{\pi}} \frac{1}{\cancel{2}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2$$

$$\boxed{Var(x) = \sigma^2}$$

put $\frac{z^2}{2} = t$, $z = \sqrt{2t}$

D.w.r.t z

$$\frac{\cancel{2}z\, dz}{\cancel{2}} = dt$$

$$z\, dz = dt$$

$$dz = \frac{dt}{z}$$

$$dz = \frac{dt}{\sqrt{2t}}$$

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n\Gamma n$$

$$\Gamma 1/2 = \sqrt{\pi}$$

$$\int_0^{\infty} e^{-x^2/2} dx = \sqrt{\frac{\pi}{2}}$$

$$\int_0^{\infty} e^{-mx} x^{p-1} dx = \frac{\Gamma p}{m^p}$$

* QD : MD : SD is 10 : 12 : 15.

For a normal distribution, the three important measures of dispersion — Quartile deviation (QD), Mean deviation (MD), and standard deviation (SD) have a fixed approximate relationship.

* Quartile deviation (QD) $= \frac{Q_3 - Q_1}{2}$

* Mean deviation (MD) $= \frac{1}{n} \sum |x - \bar{x}|$

* Standard deviation (SD) $= \sqrt{\frac{1}{n} \sum (x - \bar{x})^2}$

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In the case of normal distribution

$QD \simeq 0.675\sigma$ approximately $\frac{2}{3}$ of the standard deviation (σ)

$MD \simeq 0.797\sigma$ $\frac{4}{5}\sigma$

$SD \simeq \sigma$

Dividing all values by 0.0675 (or simplifying proportionately), the ratio becomes: $QD:MD:SD = \frac{2}{3}\sigma : \frac{4}{5}\sigma : 1\sigma$

$QD:MD:SD = 10:12:15$ $= \left(\frac{2}{3} \times 15\right) : \left(\frac{4}{5} \times 15\right) : 1 \times 15$

<u>Conclusion</u>: $= 10:12:15$

Thus, for a normal distribution, the measures of dispersion stand in the ratio:

$QD:MD:SD = 10:12:15$

This shows that Quartile deviation is the least, Mean Deviation is moderate, and standard deviation is the greatest measure of dispersion.